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General Relativity without Tensors in a Central Field

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Abstract

For static and spherically symmetric gravitational fields in the general theory of relativity, it is found possible completely to avoid tensor analysis. The principle of equivalence, illustrated by Einstein's elevator, is used to obtain Schwarzschild's equation, on which the three well-known tests of the general theory are usually based. The derivation is guided, as with Einstein, by Poisson's (Laplace's, in empty space) equation, which here can be solved by simple calculus.

1. *Introduction*

For a central stationary gravitational field, a derivation of Schwarzschild's equation is here given without the use of tensor analysis. Many authors imply that tensors (or spinors) are necessary for the derivation of this important equation with its three well-known tests of the theory. It seems, then, that the present treatment should aid in understanding the role of the fundamental hypotheses, while making it possible for a larger number of students to become acquainted with the theory.

The method first studies the effect of an observer's own acceleration on his measurements of time and distance. Then the principle of equivalence is used to interpret the results in terms of measurements in the gravitational field. Finally, application is made of the form of Laplace's equation appropriate to the non-Euclidean metric obtained for the field.

Many others have attempted to explain one or more of the three tests by using methods other than Einstein's. Some, in particular, have attempted to obtain Schwarzschild's equation by using the principle of equivalence, but without using tensors. They have not been successful, however, some of them partly because of their method of applying the principle.[1]

[1] They indicate that an observer falling from infinity, where he is at rest, carries his metric with him. Actually, Schwarzschild's equation, from which measurements of time may be determined, contradicts the idea (see Cohn, 1969, who gives a number of references).

2. *The Principle of Equivalence*

Einstein's principle of equivalence is based on the equality of gravitational and inertial mass, and is sometimes illustrated by the equal accelerations of heavy and light objects falling side by side in a vacuum. It geometrizes physics, so that each particle in a gravitational field follows a geodesic, the straightest possible path in space-time, the path thus being determined by the geometry of space-time itself. The motion, then, is not influenced by the particle's mass, as long as that mass is so small that it generates a negligible gravitational field of its own.

If we say no more, we have described only the "weak" principle of equivalence; but we shall also need the "strong" principle (Witten, 1962), which many have explained in terms of Einstein's elevator (see, for instance, Bergmann, 1942), an enclosure similar to the space capsules of the present day, and subject to the same gravitational phenomena.

If the cable on such an elevator is cut so that it falls freely in the earth's gravitational field, an enclosed observer cannot distinguish his situation from that which would hold if he were far out in interstellar space with no acceleration. If the elevator were in interstellar space and were subjected to a force producing the acceleration of gravity,[2] the man would observe the same phenomena as he would if he were stationary on the earth's surface.

It is pointed out, however, that effects of gravitation and acceleration are not quite the same for large elevators. The direction of pull of gravitation, being toward the earth's center, is different for points on opposite sides, while no such difference would be observed by the man being accelerated in outer space. Further, because of the inverse square law, the gravitational field is stronger on the floor than on the ceiling. The principle must then ignore such "tidal" effects, or effects of "higher order". The elevators must be small.

We shall here be primarily interested, just as was Einstein in his tensor treatment, in the metric of space-time, and we shall represent the gravitational field in terms of this metric, with the field arising from the fact that time and distance measurements are different for different points in space. Space and time will be liberally sprinkled with all of the standard clocks and rods that may be needed, graduated as finely as desired.

Each observer is considered to be of negligible size (his elevator is small) so that he makes all observations at a moving or stationary point. The principle of equivalence requires local rates of change with respect to distance of his time-rate and distance measurements to be the same as the corresponding rates, with respect to the same distance, for an equivalent observer. If these corresponding rates are different, then, by definition, the observers are not equivalent.

In particular, an observer F at a point near the center of a large falling elevator might, because of higher-order effects related to the inverse square

[2] In terms of natural measurements of space and time at that point, which we shall see are slightly different from those on earth.

law, find that two identical clocks, one at his own location and the other on the floor, run at different rates. He must, however, still find that the rate of change with respect to either horizontal or vertical distance, at his own location, of the time rates of such clocks is zero, just as for the equivalent inertial observer U in the special theory of relativity.

In other words, F may (and, with the inverse square law, actually does) find the time-rate to have a relative maximum at his own height, while U finds it to be a constant function of distance. At points above, below, or located horizontally from his own position, F thus need not find the gradient of the time-rate to be zero, although other freely falling observers at such points (with their slightly different gravitational accelerations) must disagree with him.

We shall also be interested in the equivalence of a stationary observer G on a tower and a second observer A who is accelerated with respect to an inertial frame of the special theory.

Our method, in applying the principle, will consist of two steps. First, we shall obtain equations (3.3) and (3.6), which tell us how the measurements made by an accelerated observer A in an inertial frame of the special theory differ from those of an unaccelerated one, U. We shall then combine (3.6) with our principle to find requirements that must be met by the gravitational field, where a stationary observer G in the field is to be equivalent to the accelerated observer A in the special theory.

3. *Acceleration in the Special Theory*

In Figure 1, in an inertial frame of the special theory of relativity, suppose that an observer A has an acceleration in the direction of increasing x, and is stationary at $t = x = y = 0$ (the origin), where t, x, y, and z are the measurements of time and distance made by an observer U who is always stationary in the frame. The acceleration is to be a constant function of time according to A (so that he can be equivalent to the observer G already mentioned), although, since we are dealing with relativity, not according to U's observations.

Take the variable τ, at A's position, to be his proper time, with $\tau = 0$ at the origin. With $\tau = \tau_1$ and $t = t_1$ at $x = x_1$ (the point A_1 in Figure 1), suppose him to send a light-pulse, radar fashion, in the direction of increasing x. It is reflected at P, whose coordinates, in equations, we call simply x and t, returning to him at A_2, where $x = x_2$, $t = t_2$, and $\tau = \tau_2$. Since the velocity of light is constant, A_1P and A_2P are straight lines. Their equations will be used below. If the pulse is sent in the opposite direction, t_1 and t_2 are interchanged, as are x_1 and x_2, and τ_1 and τ_2.

With c as the velocity of light, A defines time τ at P and the corresponding distance X from A to P as

$$\tau = \tfrac{1}{2}(\tau_1 + \tau_2) \tag{3.1}$$

and

$$X = \tfrac{1}{2}c(\tau_2 - \tau_1) \tag{3.2}$$

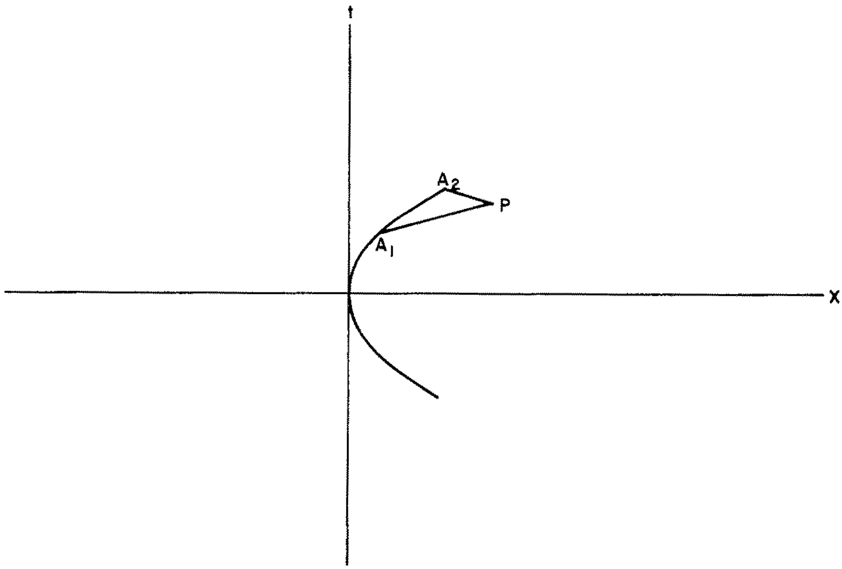

Fig. 1. Accelerated observer A sends light pulse to P and receives reflection.

just as he would have the right to do if his velocity were constant[3] or, neglecting higher-order effects, if he were falling freely.

We are led to equation (3.1) for the definition of simultaneity for both A and his equivalent observer G by the symmetry with respect to any instant of time in each case. For G, in particular, we only suppose that the light-pulse moves equally rapidly in opposite directions. From the equivalence, we need not add any further argument based on the view that A takes of the situation. The two observers must always agree, except for higher-order effects, just as must U and his equivalent freely falling observer F. Equation (3.1) is a convenient definition, not a hypothesis of the theory. We later consider other definitions to show that they lead to the same results.

It will be convenient to postpone further discussion of our definition of distance X in (3.2) until after relevant qualitative discussions which follow the derivation of (3.6). We shall also need (3.4) and (3.5).

As in the case of constant velocity in the special theory, there is motion here only along the x axis, and no change in transverse distance.

Since $x - x_1 = c(t - t_1)$ and $x - x_2 = -c(t - t_2)$ from the equations for A_1P and A_2P,

$$t = \tfrac{1}{2}(t_1 + t_2) + \tfrac{1}{2}(x_2 - x_1)/c$$

and

$$x = \tfrac{1}{2}(x_1 + x_2) + \tfrac{1}{2}c(t_2 - t_1)$$

[3] For constant velocity, the Lorentz equations can be derived from these definitions, with the aid only of the correct expression for proper time.

The variable x_1 is a function of t_1, which in turn is a function of τ_1; and similarly for x_2, t_2, and τ_2. Treating τ_1 and τ_2 as a pair of independent variables, with X and τ as another such pair, and holding X constant, we may then write

$$\partial t/\partial\tau = (\partial t/\partial t_1)(dt_1/d\tau_1)(\partial\tau_1/\partial\tau) + (\partial t/\partial t_2)(dt_2/d\tau_2)(\partial\tau_2/\partial\tau).$$

From (3.1) and (3.2),

$$\tau_1 = \tau - X/c$$

and

$$\tau_2 = \tau + X/c$$

so that, indicating derivatives of x_1 and x_2 with respect to t_1 and t_2 by dots,

$$\partial t/\partial\tau = \tfrac{1}{2}(1 - \dot{x}_1/c)(dt_1/d\tau_1) + \tfrac{1}{2}(1 + \dot{x}_2/c)(dt_2/d\tau_2)$$

For τ constant, we have

$$\begin{aligned}\partial x/\partial X &= (\partial x/\partial t_1)(dt_1/d\tau_1)(\partial\tau_1/\partial X) + (\partial x/\partial t_2)(dt_2/d\tau_2)(\partial\tau_2/\partial X)\\ &= \tfrac{1}{2}(-c + \dot{x}_1)(dt_1/d\tau_1)(-1/c) + \tfrac{1}{2}(c + \dot{x}_2)(dt_2/d\tau_2)(1/c)\end{aligned}$$

from which we immediately conclude that

$$\partial t/\partial\tau = \partial x/\partial X \tag{3.3}$$

The two left members of equation (3.6), which we now develop, follow immediately. The presence of the right member is of interest to us primarily because it tells us that the others have the same sign as that of the acceleration.

Since, from the special theory,

$$\begin{aligned}d\tau_1/dt_1 &= (1 - \dot{x}_1^{\,2}/c^2)^{1/2}\\ dt_1/d\tau_1 &= 1/(d\tau_1/dt_1) = 1\end{aligned} \tag{3.4}$$

at the origin, while

$$\begin{aligned}d^2t_1/d\tau_1^{\,2} &= (d/dt_1)(d\tau_1/dt_1)^{-1}(dt_1/d\tau_1)\\ &= -(d^2\tau_1/dt_1^{\,2})(dt_1/d\tau_1)^3\\ &= (1 - x_1^{\,2}/c^2)^{-1/2}\dot{x}_1\ddot{x}_1(dt_1/d\tau_1)^3/c^2\end{aligned}$$

so that

$$d^2t_1/d\tau_1^{\,2} = 0 \tag{3.5}$$

at the origin. The similar statements are true for the derivatives of t_2 with respect to τ_2.

Further calculation gives

$$(\partial/\partial X)(\partial t/\partial \tau) = \tfrac{1}{2}\,[\ddot{x}_1(dt_1/d\tau_1)^2 + \ddot{x}_2(dt_2/d\tau_2)^2]/c^2$$
$$+ \tfrac{1}{2}(1 - \dot{x}_1/c)(d^2t_1/d\tau_1^2)(-1/c)$$
$$+ \tfrac{1}{2}(1 + \dot{x}_2/c)(d^2t_2/d\tau_2^2)(1/c)$$

At the origin, then, from (3.3),

$$(\partial/\partial X)(\partial t/\partial \tau) = (\partial/\partial X)(\partial x/\partial X) = a/c^2 \tag{3.6}$$

where $a = \ddot{x}_1 = \ddot{x}_2$ at the origin, the acceleration of A there.

4. *Qualitative Considerations*

Equation (3.6) apparently does not occur elsewhere in the literature. Qualitatively, however, it should not have been difficult to predict that we would find that the left members must be positive for positive values of a. First, those familiar with the special theory could, without the above calculation, suspect from the Lorentz equation

$$\tau[1 - (v^2/c^2)]^{1/2} = t - vx/c^2$$

that something of the sort might be true. They know that, for constant velocities, if a railroad train moves first to the left and then to the right on a track, and if clocks next to the track all give the correct time for a stationary frame of reference those clocks toward the right must first seem to the train's passengers to be behind and then ahead of the others. The passengers might then judge that such a change in relative velocity makes the clocks to the right run faster.

The same result is suggested by Figure 1 if we ignore the difference between t_1 and τ_1, and that between t_2 and τ_2, as we may when near the origin. For P below the x axis, A_1P is less than A_2P in length, so that its projection on the t axis is also less than that of A_2P. At a later time when P is above the axis, A_1P becomes greater than A_2P. A, however, ignores both of these facts in ascribing the value $\tfrac{1}{2}(\tau_1 + \tau_2)$ to τ at P, thus supposing that the time intervals connected with A_1P and A_2P are always equal. His supposition produces the same effect as was noted for the passengers on the train: For P to the right, t is first less than and then greater than τ, and the acceleration thus causes clocks to the right to run faster.

The distance effect may be seen qualitatively in Figure 2. If the points A_{-2}, A_{-1}, A_0, A_1, and A_2 on the curve are near the origin and are equally spaced with respect to time τ, they are, except for higher-order effects, also equally spaced with respect to t. We are justified in making such a statement, because we have already, in (3.4) and (3.5), seen something of the relation between t and τ on the curve near the origin. But the points C', B', A_0, B, and C, on the x axis are not equally spaced. Instead, the distances between them increase as we move to the right. Distance x, measured by U, thus increases more rapidly than A, who compares it with X, believes it should.

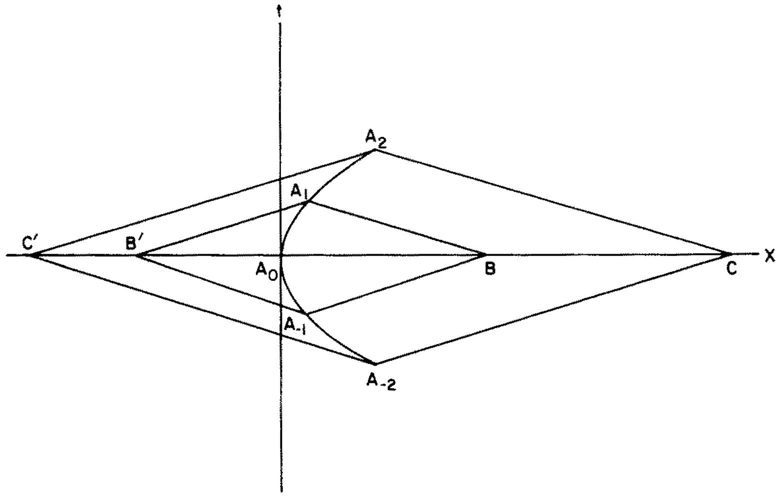

Fig. 2. *A* sends pulses to points at different distances.

It will be useful to note that, according to U, A is measuring only a part of a positive distance x; so that, for the point B, X is OB minus the x coordinate of A_1 (again neglecting higher-order effects). Similarly, for B', X is the negative of the sum of the same x coordinate of A_1 and the distance OB'. For A, U is measuring too much when x is positive. The two there then agree that X is less than x, because X and x are the measures of different distances.

They also agree, quantitatively, on equation (3.2). Its justification (as a reasonable definition), the completion of which we postponed, may now be made to follow from (3.4) and (3.5). From symmetry with respect to the x axis, U finds that A measures distance along the axis by sending a light pulse over a distance of $-ct_1 = ct_2 = \frac{1}{2}c(t_2 - t_1)$ and back. We may then write the quantity $\partial^2 x/\partial X^2$ in equation (3.6) at the origin as $(1/c^2)d^2x/dt_2{}^2$, which in turn may be replaced by $(1/c^2)d^2x/d\tau_2{}^2$. It then makes no difference whether we use $\frac{1}{2}c(t_2 - t_1)$ or $\frac{1}{2}c(\tau_2 - \tau_1)$ for our derivatives at the origin. For other points, the proper time is fundamental for A, so it must be used, as it is in (3.2).

The same result may be obtained by comparing x and X from the viewpoint of A or G, but the proof is complicated by the time effect. Time-rate registered by standard clocks, and consequently the velocity of light in both transverse and logitudinal directions, must be found by A, in terms of his own time-rate, to be multiplied by the factor $\partial t/\partial\tau$. In his comparison of x and X, he then may eliminate this time effect by dividing both time-rate and light velocity by the same factor, taking the velocity of light once again to be the constant c, just as it was for U. U's result follows.

5. *The Gravitational Field and Spherical Polar Coordinates*

The gravitational field acting on the observer G on the tower must, as we have said, be equivalent to the inertial field that A, accelerated upward, experiences. He must then find that the field has increased the time-rate above his position and decreased it below, while decreasing the vertical distance above and increasing it below. The rates of change of these quantities are related according to equation (3.6).

For points on the x axis near G, for all G, we now introduce our gravitational field in spherical polar coordinates by changing the equation for generalized distance,

$$ds^2 = c^2dt^2 - dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.1}$$

familiar in the special theory, to

$$ds^2 = C^2dt^2 - w^2dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.2}$$

where C and w are to be variable functions of r.

We here are taking $r = 0$ at the center of the spherical gravitating mass (the sun or earth). The effect of the field becomes negligible for large r, so that (5.2) may there be replaced by (5.1), thus making $C = c$ and $w = 1$ in interstellar space, far from the earth or sun.

We have changed our notation, so that t does not have the same meaning as it did in equation (3.6), and the reader should keep the change in mind until the end of the section. Its differential, dt, is now the differential of proper time (the time-rate of his standard clock) for a stationary observer S in interstellar space.

If the units of measurement are chosen so that $c = 1$, ds, in (5.1) or (5.2), is the differential of a proper time as observed by S. For other units, in order to avoid expressing time in terms of units of distance, we divide by c, so that proper time is expressed by the differential ds/c. For zero velocity, the differentials of space coordinates are zero, so that, in (5.2), $ds/c = Cdt/c$ is the differential of proper time.

The quantity r, from the manner in which r^2 occurs as a coefficient in (5.2), must be the quotient of the circumference of a central circle by 2π. The values of θ and ϕ may similarly be determined by the metric on a central sphere. All of these measurements must be made by the observer S or another unaccelerated observer G.

At G (on a tower of indefinite height), we take $r = r_1$, $C = C_1$, and $w = w_1$. We identify the x axis with the radial line through G, where $x = 0$. The differential of proper time for G is now seen to be $C_1 dt/c$, so that he finds the proper time rate for others to be the quotient $(Cdt/c)/(C_1 dt/c) = C/C_1$. These others must be stationary, whether they are in free fall at the moment or with no acceleration.

In the earlier notation of equations (3.3) and (3.6), the proper time rate for other stationary clocks is judged by A (when he is momentarily stationary) as $\partial t/\partial \tau$. Taking A and G as equivalent, we then identify $\partial t/\partial \tau$ with C/C_1 in (3.6).

We next identify dX, the differential of distance on the axis as judged by A, with wdr, the differential of radial distance as judged by G. We shall now see that it follows that dx must be identified with $w_1 dr$. First, $wdr/(w_1 dr) = 1$ at G, just as $\partial x/\partial X = 1$ at $x = 0$ on the x axis. Further, just as C_1/c, F's time rate with respect to that in (5.1) (before the imposition of the gravitational field), has a gradient of zero, so must the corresponding rate of radial measurements, so it is the constant w_1. In each case, according to the principle of equivalence, the gradient produced by the field for G is nullified for F by their relative acceleration.

From our replacements in (3.6), we thus have the equation

$$(d/dr)(C/C_1) = (d/dr)(w_1/w)$$

for $r = r_1$. From simple calculus, this becomes

$$(dC/dr)/C_1 = -w_1(dw/dr)/w^2$$

At G, where $r = r_1$, this may be written

$$(dC/dr)_1/C_1 = -(dw/dr)_1/w_1$$

Since r_1 is arbitrary, as long as G is in empty space,

$$(dC/dr)/C = -(dw/dr)/w \tag{5.3}$$

Integration gives

$$\ln C = -\ln w + \text{const}$$

and, since $C = c$ and $w = 1$ in interstellar space,

$$\ln C = -\ln w + \ln c = \ln(c/w)$$

It follows that

$$C = c/w, \qquad \text{or} \qquad w = c/C \tag{5.4}$$

In equation (5.2) of this section, we have made the assumption, as in (3.1), that the velocity of light is equal in opposite radial directions, as is apparent if we take $ds = 0$ to study the behavior of light. It is well known in the literature (Tolman, 1934) that the *drdt* term otherwise obtained may be eliminated by a substitution for the time t. There is no fundamental difficulty, then, with our definition expressed by (3.1).

6. *The Energy Equation and the Law of Areas*

In the special theory, a particle subject to no force follows a straight line with $\int ds$ an extremum. In (5.2), we continue to make the integral an extremum, as required by the weak principle of equivalence, thus taking the path as near to a straight line in our new metric as possible.

From the calculus of variations, we then have (Akhiezer, 1962)

$$C^2\, dt/ds = \text{const} \tag{6.1}$$

the quantity on the left side being obtained by taking the partial derivative of

$$C^2\,(dt/ds)^2 - w^2\,(dr/ds)^2 - r^2\,[(d\theta/ds)^2 + \sin^2\theta(d\phi/ds)^2]$$

with respect to dt/ds. This is permitted because of the independence of the coefficients of (5.2) of t.

Similarly, restricting ourselves to plane motion by taking $\theta = \frac{1}{2}\pi$, we have, noting that the coefficients are independent of ϕ as well as of t,

$$r^2\, d\phi/ds = \text{const} \tag{6.2}$$

the law of areas in general relativity.

The similar quantities for rectangular coordinates in the special theory, $c^2 dt/ds$, dx/ds, dy/ds, and dz/ds, when multiplied by $m_0 c$, where m_0 is the rest mass of a body in interstellar space, are the energy and the rectangular components of the momentum. If we also multiply the quantities in (6.1) and (6.2) by $m_0 c$, it is natural to call the first product the energy and the second the angular momentum. Indeed, for a particle moving from interstellar space, where $C = c$, into the gravitational field of the sun, the two expressions for energy are equal.

We thus have the equation for the energy E of a small mass (small, because it is only the central gravitational field which we are considering),

$$m_0 c C^2 dt/ds = m_0 c C/(1 - v^2/C^2)^{1/2} = E \tag{6.3}$$

where

$$v^2 = (w dr/dt)^2 + r^2\,[(d\theta/dt)^2 + \sin^2\theta(d\phi/dt)^2]$$

and where it is understood that the energy is being judged by a stationary observer S in interstellar space.

Equation (6.3) is a rather remarkable formula, which, in spite of its simplicity, apparently does not occur in the literature. We have actually proved it only for v numerically greater than the escape velocity, so as to make use of the situation near the observer S in interstellar space. However, the formula must also hold for smaller v, as we shall now see.

We have already noticed that the differential of proper time for a stationary observer in the field is Cdt/c. It follows that, if t is to be replaced by this proper time, the velocity v must be replaced by $u = v/(C/c)$, so that $v/C = u/c$.

The special theory holds in the immediate space-time neighborhood of a stationary observer G in the field, and he finds the energy of a small mass there to be

$$m_1 c^2/(1 - u^2/c^2)^{1/2} = m_1 c^2/(1 - v^2/C^2)^{1/2}$$

where m_1 is G's measurement of the mass which would be m_0 at S.

The amount of energy emitted by a change in velocity from v_1 to v_2, at G and according to G, is

$$m_1c^2/(1 - v_1^2/C^2)^{1/2} - m_1c^2/(1 - v_2^2/C^2)^{1/2}$$

and similarly for v_3 and v_4. If these two emitted energies are equal, we may multiply each by $(m_0/m_1)(C/c)$ to obtain the equation

$$m_0cC[1/(1 - v_1^2/C^2)^{1/2} - 1/(1 - v_2^2/C^2)^{1/2}]$$
$$= m_0cC[1/(1 - v_3^2/C^2)^{1/2} - 1/(1 - v_4^2/C^2)^{1/2}]$$

For sufficiently large v_1 and v_2, the left member is an emitted energy according to observer S. The right side, then, must be the same, regardless of the magnitudes of v_3 and v_4. Assuming only that v_1, v_2, and v_3 are greater than the velocity of escape (as we may for all possible v_4), it follows that

$$m_0cC/(1 - v_4^2/C^2)^{1/2}$$

is the energy, according to S, even for small v_4.

It is then possible to take $v = 0$ in (6.3). Doing so, we obtain the potential energy m_0cC.

7. *The Laplacian in the Non-Euclidean Metric*

Einstein, in his tensor treatment, looked for "an analogue of Poisson's equation" (Laplace's in empty space). In his earlier work, he used a variational principle (Einstein et al., 1973), while he later employed different methods for arriving at his analog (Einstein, 1956).

For those who are familiar with the many applications of Poisson's (or Laplace's) equation to problems in electrical potential, magnetic potential, the flow of fluids, the flow of heat, etc., it is not surprising that Einstein would think of it.

For the classical gravitational field, Laplace's equation emerges from the idea that the total gravitational flux into any region of empty space is zero, while Poisson's equation gives the flux into a region which may be occupied by masses.

For our own non-Euclidean metric, the flux across the surface of a sphere of surface area $4\pi r^2$, concentric with the sun, is proportional to the product of the area and $(1/w)dC/dr$, the rate of change of the potential with respect to distance as given by standard rods, perpendicular to the surface. This quantity must be constant, for otherwise there would be a net flux into the region between two such surfaces.[4] We then have

$$(1/w)r^2dC/dr = \text{const} \qquad (7.1)$$

[4] Probably the simplest application of Laplace's equation is to temperature C. If the heat content of no region changes with time, the equation is satisfied.

The actual Laplacian for a space metric

$$d\sigma^2 = h_1{}^2 du_1{}^2 + h_2{}^2 du_2{}^2 + h_3{}^2 du_3{}^2$$

is (Arfken, 1968)

$$[(\partial/\partial u_1)(h_2 h_3 h_1^{-1}\, \partial C/\partial u_1) + (\partial/\partial u_2)(h_3 h_1 h_2^{-1}\, \partial C/\partial u_2) + (\partial/\partial u_3)(h_1 h_2 h_3^{-1}\, \partial C/\partial u_3)]/(h_1 h_2 h_3)$$

which, for us, gives the Laplace equation

$$(1/w)(d/dr)[(r^2/w)dC/dr]/r^2 = 0 \tag{7.2}$$

Integration of (7.2) again gives (7.1).

Our equating the Laplacian to zero may be said to parallel Einstein's setting a certain tensor equal to zero in empty space.

Although it is also possible to obtain (7.1) by applying Hamilton's principle to the energy of the gravitational field, we shall not do so. Any such development, providing a bridge between mathematical equations and the physical world, must necessarily involve assumptions.

8. *Schwarzschild's Equation*

From (7.1), (5.4), and (5.2), there must be a constant k for which

$$r^2 C dC/dr = k \tag{8.1}$$

and consequently

$$C^2 = c^2 - 2k/r \tag{8.2}$$

Schwarzschild's equation,

$$ds^2 = c^2(1 - 2k/c^2 r)dt^2 - dr^2/(1 - 2k/c^2 r) - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{8.3}$$

follows from (5.2), (5.4), and (8.2).

The inverse square law, in the form

$$d^2r/ds^2 = -k/c^2 r^2$$

may be obtained from (8.3) and the energy equation if it is assumed that both $d\theta/ds$ and $d\phi/ds$ are zero. It is also possible to start with this law and obtain (8.3) without Laplace's equation.

The three well-known tests of the general theory, discussed in many books on the subject (Bergmann, 1942, pp. 212–222), spring from Schwarzschild's equation. With the aid of the energy equation and the law of areas, it may be used to find the orbit of a planet around the sun, and thus leads to the motion of the perihelion of that orbit. The value of C alone, in the equation, leads to the red shift of spectral lines in the gravitational field. The bending of light as it passes the sun is obtainable from the equation with the aid of the appropriate form of the law of areas. We here must take $ds = 0$ and $\int dt$ a minimum to find that

$$r^2 d\phi/(1 - 2k/c^2 r)dt = \text{const}$$

Conclusion

We have seen that the principle of equivalence demands a relation between time and distance measurements indicated, for a central field, by (5.4). We have then found reason to define empty space, mathematically, as that region where the Laplacian, in our metric, of the gravitational potential is zero, just as Einstein took one of his tensors to be zero there. We have not even mentioned the principle of general covariance, often thought to be one of the main foundation stones of the theory (Hoffmann, 1972).

Max Born, in a letter to Einstein, once spoke of "the horrible complications of the formalism" of the general theory (Born, 1971). The present treatment has been an effort to avoid some of these and to provide new insight into the structure of the theory.

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